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2003 J. Phys. A: Math. Gen. 36 4937

(http://iopscience.iop.org/0305-4470/36/17/319)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4937–4945 PII: S0305-4470(03)59016-3

Lorentz and CPT symmetries in commutative and noncommutative spacetimes

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Received 30 January 2003, in final form 7 March 2003
Published 16 April 2003
Online at stacks.iop.org/JPhysA/36/4937

Abstract

We investigate the fermionic sector of a given theory, in which massive and charged Dirac fermions interact with an Abelian gauge field, including a non-standard contribution that violates both Lorentz and CPT symmetries. We offer an explicit calculation in which the radiative corrections due to the fermions seem to generate a Chern–Simons-like effective action. Our results are obtained under the general guidance of dimensional regularization, and they show that there is no room for Lorentz and CPT violation in both commutative and noncommutative spacetimes.

PACS numbers: 11.30.Cp, 11.303.Er

1. Introduction

Maxwell's theory of electromagnetism was crucial to question Galilei invariance, to give rise to Lorentz symmetry. Nowadays, in string theory one may find a way to question Lorentz invariance, since there are interactions that support spontaneous breaking of Lorentz symmetry [1]. In string theory, one may also find room for noncommutativity of the coordinates that define the spacetime manifold [2]. Thus, it appears legitimate to investigate possible breaking of Lorentz invariance in both commutative and noncommutative spacetimes.

The issue of breaking Lorentz invariance has been recently addressed by many authors. The standard route [3–6] includes a modification of Maxwell's theory, in which one adds the Chern–Simons-like term $\kappa_\mu \varepsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} A_\rho$. The problem relies on recognizing that Lorentz and CPT symmetries are violated in the fermionic sector of a given theory, which contains the contributions [7–19]

$$I_f = \int \mathrm{d}^4 x\, \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m - A\!\!\!/ - b\!\!\!/\gamma_5)\psi. \tag{1}$$

The first three terms are usual; they describe charged and massive Dirac fermions coupled to an Abelian gauge field. However, the fourth term is unusual: b_μ is a constant 4-vector which selects a fixed direction in spacetime, and explicitly violates Lorentz and CPT symmetries. The fermions can be integrated, and the radiative result may lead to

$$I_{\mathrm{CS}} = \frac{1}{2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\lambda\rho}\kappa_\mu F_{\nu\lambda}A_\rho \tag{2}$$

with κ_μ being proportional to b_μ, that is $\kappa_\mu = Cb_\mu$. This result, if correct, introduces a modification of electrodynamics, which allows for the explicit violation of Lorentz and CPT symmetries. The issue has been carefully investigated in several different contexts, leading to results in which C vanishes [7, 14] or those in which it does not [8–13, 15–19].

In the present work we revisit the problem, with the aim of extending the calculation to the noncommutative spacetime manifold. The importance of investigating the noncommutativity of spacetime has been brought to high-energy physics via string theory [2] (see also [20, 21] for other information). In our quest to deal with the issue in the standard case, however, we had to introduce new calculations which led us to the result that there is neither Lorentz nor CPT violation in the commutative spacetime. And this was also shown to be correct in the noncommutative case. These results were obtained under the general guidance of dimensional regularization, and they have led us to offer our calculations in a form as standard as possible, keeping track of the main steps and enlightening the way the puzzle shows up: as we shall see, there is an intricate entanglement between the calculation involving the Dirac matrices and the evaluation of the momentum integral of all the contributions at first order in the vector that responds for Lorentz and CPT violation, and at second and third (in the noncommutative case) orders in the gauge field. We implement our investigations using the derivative expansion of operators [22–26], and we consider the spacetime as commutative and noncommutative.

2. Commutative spacetime

Firstly we work in the commutative case. To account for the fermionic integration we write

$$\mathrm{e}^{\mathrm{i}I[b,A]} = \int D\bar{\psi}\,D\psi\;\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_f} \tag{3}$$

where the effective action is given by

$$I[b, A] = -\mathrm{i}\,\mathrm{Tr}\ln(\not{p} - m - \not{A} - \not{b}\gamma_5). \tag{4}$$

We use this expression to write: $I[b, A] = I[b] + I'[b, A]$. The first term is $I[b] = -\mathrm{i}\,\mathrm{Tr}\ln(\not{p} - m - \not{b}\gamma_5)$, which does not depend on the gauge field. The second term is $I'[b, A]$, which is given by

$$I'[b, A] = \mathrm{i}\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{1}{\not{p} - m - \not{b}\gamma_5}\not{A}\right]^n. \tag{5}$$

In this expression we single out the term

$$I^{(2)}[b, A] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\,\frac{1}{\not{p} - m - \not{b}\gamma_5}\not{A}\frac{1}{\not{p} - m - \not{b}\gamma_5}\not{A} \tag{6}$$

which is the term that matters, in the quest to find how the radiative corrections generate the Chern–Simons-like term written in equation (2).

We can proceed following two distinct routes, in which one includes or not the contribution involving the vector b_μ in the Dirac propagator (see [8] for details). In the present work we follow the perturbative route, so we use the expression

$$\frac{1}{\not{p} - m - \not{b}\gamma_5} = \frac{1}{\not{p} - m} + \frac{1}{\not{p} - m}\not{b}\gamma_5\frac{1}{\not{p} - m} + \cdots \tag{7}$$

to write, to first order in b and second order in A

$$I^{(1,2)}[b, A] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}[S(p)\not{b}\gamma_5 S(p)\not{A}S(p)\not{A} + S(p)\not{A}S(p)\,\not{b}\gamma_5 S(p)\not{A}] \tag{8}$$

where we have set

$$S(p) = \frac{1}{\not{p} - m}. \tag{9}$$

We rewrite equation (8) in the form

$$I^{(1,2)}[b, A] = \frac{\mathrm{i}}{2}\int \mathrm{d}^4x\left(\Pi_a^{\mu\nu} + \Pi_b^{\mu\nu}\right)A_\mu A_\nu \tag{10}$$

where

$$\Pi_a^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\not{b}\gamma_5 S(p)\gamma^\mu S(p - \mathrm{i}\partial)\gamma^\nu \tag{11}$$

and

$$\Pi_b^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\gamma^\mu S(p - \mathrm{i}\partial)\not{b}\gamma_5 S(p - \mathrm{i}\partial)\gamma^\nu \tag{12}$$

where tr stands for the trace over the Dirac matrices.

We now follow [22–26] and use the expansion

$$\frac{1}{\not{p} - \mathrm{i}\not{\partial} - m} = \frac{1}{\not{p} - m} + \frac{1}{\not{p} - m}\mathrm{i}\not{\partial}\frac{1}{\not{p} - m} + \cdots \tag{13}$$

which is valid up to first order in ∂, which is the expression we need to generate the Chern–Simons-like term. With this we change $\Pi_a^{\mu\nu} \to \Pi_1^{\mu\nu}$ and rewrite it in the form

$$\Pi_1^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\not{b}\gamma_5 S(p)\gamma^\mu S(p)\mathrm{i}\not{\partial} S(p)\gamma^\nu. \tag{14}$$

Also, we change $\Pi_b^{\mu\nu} \to \Pi_2^{\mu\nu} + \Pi_3^{\mu\nu}$ to write

$$\Pi_2^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\gamma^\mu S(p)\not{b}\gamma_5 S(p)\mathrm{i}\not{\partial} S(p)\gamma^\nu \tag{15}$$

and

$$\Pi_3^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\gamma^\mu S(p)\mathrm{i}\not{\partial} S(p)\not{b}\gamma_5 S(p)\gamma^\nu. \tag{16}$$

We work with $\Pi_1^{\mu\nu}$. It can be written as $\Pi_1^{\mu\nu} = \Pi_{1,\mathrm{div}}^{\mu\nu} + \Pi_{1,\mathrm{fin}}^{\mu\nu}$, where

$$\Pi_{1,\mathrm{div}}^{\mu\nu} = \mathrm{i}b_\lambda\,\mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\not{p}\gamma^\lambda\gamma_5\not{p}\gamma^\mu\not{p}\not{\partial}\not{p}\gamma^\nu}{(p^2 - m^2)^4} \tag{17}$$

and

$$\begin{aligned}\Pi_{1,\mathrm{fin}}^{\mu\nu} = \mathrm{i}m^2 b_\lambda\,\mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2 - m^2)^4}(&\not{p}\gamma^\lambda\gamma_5\not{p}\gamma^\mu\not{\partial}\gamma^\nu + \not{p}\gamma^\lambda\gamma_5\gamma^\mu\not{p}\not{\partial}\gamma^\nu + \not{p}\gamma^\lambda\gamma_5\gamma^\mu\not{\partial}\not{p}\gamma^\nu\\ &+ \gamma^\lambda\gamma_5\not{p}\gamma^\mu\not{p}\not{\partial}\gamma^\nu + \gamma^\lambda\gamma_5\not{p}\gamma^\mu\not{\partial}\not{p}\gamma^\nu + \gamma^\lambda\gamma_5\gamma^\mu\not{p}\not{\partial}\not{p}\gamma^\nu + m^2\gamma^\lambda\gamma_5\gamma^\mu\not{\partial}\gamma^\nu).\end{aligned} \tag{18}$$

The other two terms $\Pi_2^{\mu\nu}$ and $\Pi_3^{\mu\nu}$ are similar, and are treated similarly.

We evaluate the integrals under the general guidance of dimensional regularization [27–29]. Thus, we change dimensions from 4 to $2w$, and we change $\mathrm{d}^4p/(2\pi)^4$ to $(\mu^2)^{2-w}[\mathrm{d}^{2w}p/(2\pi)^{2w}]$, where μ is an arbitrary parameter that identifies the mass scale. We use two distinct routes to perform the calculations involving the Dirac matrices. In the first route we use the cyclic property of the trace, to move γ_5 to the very end of every expression involving the trace of Dirac matrices. The potential divergences in the momentum integration come from the first term of $\Pi_1^{\mu\nu}$. We use

$$\int \frac{\mathrm{d}^{2w}p}{(2\pi)^{2w}} \frac{p_\alpha p_\beta p_\gamma p_\delta}{(p^2-m^2)^4} = \frac{\mathrm{i}}{24(4\pi)^w} \frac{\Gamma(2-w)}{(m^2)^{2-w}} G_{\alpha\beta\gamma\delta} \tag{19}$$

where $G_{\alpha\beta\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}$. We also use $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$ and $\gamma^\alpha\gamma_\alpha = 2w$ in order to rewrite equation (10) in the form

$$I^{(1,2)}[b, A] = \frac{3}{2}\mathrm{i}\Pi(w) b_\mu \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_5) \int \mathrm{d}^4x\, \partial_\nu A_\lambda A_\rho. \tag{20}$$

Here the factor 3 accounts for identical contributions that come from $\Pi_1^{\mu\nu}$, $\Pi_2^{\mu\nu}$ and $\Pi_3^{\mu\nu}$. Also, $\Pi(w)$ is given by

$$\Pi(w) = -\frac{2w-1}{96\pi^2} + \frac{w+1}{96\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-w} \Gamma(2-w)(2-w). \tag{21}$$

In the above calculations we have set $\Pi_1^{\mu\nu} = \Pi_{1,\mathrm{div}}^{\mu\nu} + \Pi_{1,\mathrm{fin}}^{\mu\nu}$ to split the $\Pi_1^{\mu\nu}$ contribution into two parts, one divergent and the other finite. The contribution $\Pi_{1,\mathrm{div}}^{\mu\nu}$ is divergent in the limit $w \to 2$, and it contributes with the term proportional to $\Gamma(2-w)$. However, the factor involving the Dirac matrices contributes with the term $(2-w)$, in such a way that the full contribution is finite in the limit $w \to 2$. Furthermore, this finite term exactly compensates the finite contribution that appears from $\Pi_{1,\mathrm{fin}}^{\mu\nu}$ in the limit $w \to 2$. In the limit $w \to 2$ we can use $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_5) = 4\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}$, but $\Pi(w \to 2) \to 0$ and this leaves no room for Lorentz and CPT violation. The perfect balance between the two contributions that we have just found has been identified before in [30] as being peculiar to dimensional regularization. We stress that if one uses the relation $\{\gamma^\mu, \gamma_5\} = 0$ to move γ_5 to the end of every expression involving the trace of Dirac matrices, the perfect balance between the two contributions is broken, giving rise to a nonzero value for the constant C. This result is due to the use of $\{\gamma^\mu, \gamma_5\} = 0$, which is valid in the four-dimensional spacetime, but we are working in $2w$ dimensions.

We make this point stronger by considering another route to implement the calculation involving properties of the Dirac matrices when the spacetime has dimension $2w$. We follow [29, 31–33] and now the Dirac matrices corresponding to the external indices μ, ν and λ are physical matrices; they are written in the form $\bar{\gamma}^\mu$, etc. The contribution

$$\operatorname{tr}(\gamma^\alpha\bar{\gamma}^\lambda\gamma_5\gamma^\beta\bar{\gamma}^\mu\gamma^\gamma\gamma^\rho\gamma^\delta\bar{\gamma}^\nu)G_{\alpha\beta\gamma\delta} \tag{22}$$

splits into the three terms

$$\operatorname{tr}(\gamma^\alpha\bar{\gamma}^\lambda\gamma_5\gamma_\alpha\bar{\gamma}^\mu\gamma^\beta\gamma^\rho\gamma_\beta\bar{\gamma}^\nu) + \operatorname{tr}(\gamma^\alpha\bar{\gamma}^\lambda\gamma_5\gamma^\beta\bar{\gamma}^\mu\gamma_\alpha\gamma^\rho\gamma_\beta\bar{\gamma}^\nu) + \operatorname{tr}(\gamma^\alpha\bar{\gamma}^\lambda\gamma_5\gamma^\beta\bar{\gamma}^\mu\gamma_\beta\gamma^\rho\gamma_\alpha\bar{\gamma}^\nu) \tag{23}$$

and the Dirac matrices are changed according to $\gamma^\alpha \to \bar{\gamma}^\alpha + \hat{\gamma}^\alpha$, where $\{\bar{\gamma}^\alpha, \bar{\gamma}^\beta\} = 2\bar{g}^{\alpha\beta}$, $\{\hat{\gamma}^\alpha, \hat{\gamma}^\beta\} = 2\hat{g}^{\alpha\beta}$, and $\{\bar{\gamma}^\alpha, \hat{\gamma}^\beta\} = 0$, and also $\bar{\gamma}^\alpha\bar{\gamma}_\alpha = 4$, $\bar{\gamma}^\alpha\hat{\gamma}_\alpha = 0$ and $\hat{\gamma}^\alpha\hat{\gamma}_\alpha = 2(w-2)$. In this case we can use either the cyclic property of the trace, or the relations $\{\gamma_5, \bar{\gamma}^\mu\} = [\gamma_5, \hat{\gamma}^\mu] = 0$ to rewrite equation (10) in the form

$$I^{(1,2)}[b, A] = \frac{3}{2}\Pi'(w) b_\mu \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\lambda\rho}\partial_\nu A_\lambda A_\rho \tag{24}$$

where we have used $\operatorname{tr}(\bar{\gamma}^\mu\bar{\gamma}^\nu\bar{\gamma}^\lambda\bar{\gamma}^\rho\gamma_5) = 4\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}$ and $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\hat{\gamma}^\rho\gamma_5) = 0$. Also, $\Pi'(w) = -4\Pi(w)$. We use this result to write equation (20) as in the Chern–Simons-like term in equation (2), where $\kappa_\mu = Cb_\mu$, with C given by

$$C = \frac{2w-1}{16\pi^2} - \frac{1+w}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-w} \Gamma(2-w)(2-w). \tag{25}$$

We see that $C \to 0$ in the limit $w \to 2$, which confirms the former result, in which we have used the cyclic property of the trace.

3. Spacetime as a noncommutative manifold

We now consider the spacetime as noncommutative [34–41]. In this case we set $[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is a constant antisymmetric tensor. As a consequence, one replaces the ordinary product of functions by the Moyal product

$$f(x) \star g(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu\partial'_\nu} f(x)g(x')|_{x'=x}. \tag{26}$$

The first modification we have to introduce concerns equation (1), which should be changed to

$$\tilde{I}_f = \int \mathrm{d}^4x\, \bar{\psi} \star (\mathrm{i}\not{\partial} - m - \not{b}\gamma_5 - \not{A}\star)\psi \tag{27}$$

or better

$$\tilde{I}_f = \int \mathrm{d}^4x\, \bar{\psi}(x)(\mathrm{i}\not{\partial}' - m - \not{b}\gamma_5 - \mathrm{e}^{\mathrm{i}\partial\times\partial'}\not{A})\psi(x')\Big|_{x'=x} \tag{28}$$

where we are working with the fundamental (or anti-fundamental) representation of the gauge group, using $\frac{1}{2}\theta^{\mu\nu}\partial_\mu\partial'_\nu = \partial \times \partial'$. In this case equation (2) should be changed to

$$\tilde{I}_{\mathrm{CS}} = \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\lambda\rho}\tilde{\kappa}_\mu \left(\partial_\nu A_\lambda \star A_\rho + \tfrac{2}{3}\mathrm{i}A_\nu \star A_\lambda \star A_\rho\right) \tag{29}$$

where $\tilde{\kappa}_\mu$ must have the form $\tilde{\kappa}_\mu = \tilde{C}b_\mu$, to include modifications coming from the noncommutativity of spacetime.

We are working with the fundamental representation of the gauge group. In this case noncommutativity seems to change the gauge field A by the new field $\tilde{A} = \mathrm{e}^{\partial\times p}A$, and this identification eases the work we have to implement, since we now see that the above modification changes equation (5) to

$$\tilde{I}'[b, A] = \mathrm{i}\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{1}{\not{p} - m - \not{b}\gamma_5}\,\mathrm{e}^{\partial\times p}\not{A}(x)\right]^n \tag{30}$$

and now we single out the term

$$\tilde{I}^{(2)}[b, A] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\left[\frac{1}{\not{p} - m - \not{b}\gamma_5}\,\mathrm{e}^{\partial\times p}\not{A}(x)\frac{1}{\not{p} - m - \not{b}\gamma_5}\,\mathrm{e}^{\partial'\times p}\not{A}(x')\right]_{x'=x} \tag{31}$$

which modifies the former calculations as follows: we rewrite equation (10) in the form

$$\tilde{I}^{(1,2)}[b, A] = \frac{\mathrm{i}}{2}\int \mathrm{d}^4x\left(\tilde{\Pi}_a^{\mu\nu} + \tilde{\Pi}_b^{\mu\nu}\right)A_\mu \star A'_\nu\Big|_{x'=x} \tag{32}$$

where the terms $\tilde{\Pi}_a^{\mu\nu}$ and $\tilde{\Pi}_b^{\mu\nu}$ are now given by

$$\tilde{\Pi}_a^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\not{b}\gamma_5 S(p)\gamma^\mu S(p - \mathrm{i}\partial)\gamma^\nu\,\mathrm{e}^{(\partial+\partial')\times p} \tag{33}$$

and

$$\tilde{\Pi}_b^{\mu\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\gamma^\mu S(p - \mathrm{i}\partial)\not{b}\gamma_5 S(p - \mathrm{i}\partial)\gamma^\nu\,\mathrm{e}^{(\partial+\partial')\times p}. \tag{34}$$

In both cases, expanding the phase factors and propagators up to first order in the derivative the result adds to zero, as in the commutative case. Thus, in the Chern–Simons-like contribution that appears in equation (29), the term proportional to $\partial_\mu A_\nu \star A_\lambda$ remains as in the former result in the commutative case.

In the noncommutative case there is another contribution, trilinear in the gauge field, which comes from equation (30) for $n = 3$. This contribution is given by

$$\tilde{I}^{(3)}[b, A] = \frac{\mathrm{i}}{3}\,\mathrm{Tr}\left[\frac{1}{\not{p} - m - \not{b}\gamma_5}\,\mathrm{e}^{\partial\times p}\not{A}(x)\frac{1}{\not{p} - m - \not{b}\gamma_5}\mathrm{e}^{\partial'\times p}\not{A}(x') \times \frac{1}{\not{p} - m - \not{b}\gamma_5}\,\mathrm{e}^{\partial''\times p}\not{A}(x'')\right]_{x''=x'=x}. \tag{35}$$

We use equation (7) to write, selecting the terms that are linear in b,

$$\begin{aligned}\tilde{I}^{(1,3)}[b, A] = \frac{\mathrm{i}}{3}\,\mathrm{Tr}\,S(p)\big[&\mathrm{e}^{\partial\times p}\not{A}(x)S(p)\,\mathrm{e}^{\partial'\times p}\not{A}(x')S(p)\not{b}\gamma_5 S(p)\\ &+\mathrm{e}^{\partial\times p}\not{A}(x)S(p)\not{b}\gamma_5 S(p)\,\mathrm{e}^{\partial'\times p}\not{A}(x')S(p)\\ &+\not{b}\gamma_5 S(p)\,\mathrm{e}^{\partial\times p}\not{A}(x)S(p)\,\mathrm{e}^{\partial'\times p}\not{A}(x')S(p)\big]\,\mathrm{e}^{\partial''\times p}\not{A}(x'')\Big|_{x''=x'=x}.\end{aligned} \tag{36}$$

We use the identity (13) to write

$$\tilde{I}^{(1,3)}[b, A] = \frac{\mathrm{i}}{3}\int \mathrm{d}^4x \sum_{n=1}^{3} \Gamma_n^{\mu\rho\nu}\,\mathrm{e}^{(\partial+\partial'+\partial'')\times p} A_\mu \star A'_\rho \star A''_\nu\Big|_{x''=x'=x} \tag{37}$$

where

$$\Gamma_1^{\mu\rho\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\not{b}\gamma_5 S(p)\gamma^\mu S(p)\gamma^\rho S(p)\gamma^\nu \tag{38}$$

and

$$\Gamma_2^{\mu\rho\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\gamma^\mu S(p)\not{b}\gamma_5 S(p)\gamma^\rho S(p)\gamma^\nu \tag{39}$$

and

$$\Gamma_3^{\mu\rho\nu} = \mathrm{tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4} S(p)\gamma^\mu S(p)\gamma^\rho S(p)\not{b}\gamma_5 S(p)\gamma^\nu. \tag{40}$$

These three terms are very similar to the three terms $\Pi_1^{\mu\nu}$, $\Pi_2^{\mu\nu}$ and $\Pi_3^{\mu\nu}$ that we have found in equations (14), (15) and (16) of the former calculation. They contribute similarly, and we can write, expanding the phase factor up to zeroth order in the derivative,

$$\tilde{I}^{(1,3)}[b, A] = \mathrm{i}\Gamma(w)\int \mathrm{d}^4x\, b_\mu \varepsilon^{\mu\nu\lambda\rho} A_\nu \star A_\lambda \star A_\rho \tag{41}$$

where $\Gamma(w) = -4\Pi(w)$ (see equation (21)). We then add $\tilde{I}^{(1,2)}[b, A]$ and $\tilde{I}^{(1,3)}[b, A]$ to write equation (29) in the form

$$\tilde{I}_{\mathrm{CS}} = C\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\lambda\rho} b_\mu \left(\partial_\nu A_\lambda \star A_\rho + \frac{2}{3}\mathrm{i}A_\nu \star A_\lambda \star A_\rho\right) \tag{42}$$

where C is given in equation (25), the same result obtained in the commutative case. Thus, in the limit $w \to 2$ there is no room for Lorentz and CPT violation also in the noncommutative case that we have just considered.

We can also work with the adjoint representation of the gauge group. In this case, in the fermionic action in equation (5) we should change A to $\tilde{A}_{ad} = (\mathrm{e}^{\partial\times p} - \mathrm{e}^{-\partial\times p})A$; also, we include an extra factor of $1/2$ in this fermionic action, in order to account for the use of Majorana spinors. The change in the gauge field is similar to the identification done in the former case, for the fundamental representation. The calculation is similar, and the procedure that we follow is: the phase factors that appear from non-planar diagrams are also expanded up to first order in the derivative, in order to maintain the original programme of searching for contributions linear in the derivative, bilinear in the gauge field and trilinear in the gauge field. Within this context, the result in equation (32) should be changed to

$$\tilde{I}_{ad}^{(1,2)}[b, A] = \frac{\mathrm{i}}{2} \int \mathrm{d}^4x \left(\tilde{\Pi}_{a,ad}^{\mu\nu} + \tilde{\Pi}_{b,ad}^{\mu\nu}\right) [A_\mu, A'_\nu]_\star \Big|_{x'=x} \tag{43}$$

where the terms $\tilde{\Pi}_{a,ad}^{\mu\nu}$ and $\tilde{\Pi}_{b,ad}^{\mu\nu}$ are now given by

$$\tilde{\Pi}_{a,ad}^{\mu\nu} = \mathrm{tr} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} S(p) \not{b}\gamma_5 S(p)\gamma^\mu S(p)\gamma^\nu (\partial \times p) \tag{44}$$

and

$$\tilde{\Pi}_{b,ad}^{\mu\nu} = \mathrm{tr} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} S(p)\gamma^\mu S(p) \not{b}\gamma_5 S(p)\gamma^\nu (\partial \times p). \tag{45}$$

We have

$$\tilde{\Pi}_{a,ad}^{\mu\nu} = b_\lambda \,\mathrm{tr} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\partial \times p}{(p^2 - m^2)^3} \left(\not{p}\gamma^\lambda\gamma_5 \not{p}\gamma^\mu \not{p}\gamma^\nu + m^2 \not{p}\gamma^\lambda\gamma_5\gamma^\mu\gamma^\nu \right). \tag{46}$$

We use dimensional regularization in order to write

$$\int \frac{\mathrm{d}^{2w} p}{(2\pi)^{2w}} \frac{p_\alpha p_\beta}{(p^2 - m^2)^3} = \frac{\mathrm{i}(1-w)}{4(4\pi)^w} \frac{\Gamma(1-w)}{(m^2)^{2-w}} g_{\alpha\beta} \tag{47}$$

and

$$\int \frac{\mathrm{d}^{2w} p}{(2\pi)^{2w}} \frac{p_\alpha p_\beta p_\gamma p_\delta}{(p^2 - m^2)^3} = -\frac{\mathrm{i}m^2}{8(4\pi)^w} \frac{\Gamma(1-w)}{(m^2)^{2-w}} G_{\alpha\beta\gamma\delta}. \tag{48}$$

We use the above results to get

$$\tilde{\Pi}_{a,ad}^{\mu\nu} = -\frac{m^2}{16\pi^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{2-w} \Gamma(1-w)(3-w)\varepsilon^{\mu\nu\lambda\rho} b_\lambda \bar{\partial}_\rho \tag{49}$$

where $\bar{\partial}_\alpha = \theta_{\alpha\beta}\partial^\beta$.

The calculation involving $\tilde{\Pi}_{b,ad}^{\mu\nu}$ is similar. It gives the result $\tilde{\Pi}_{b,ad}^{\mu\nu} = -\tilde{\Pi}_{a,ad}^{\mu\nu}$, showing that the contribution bilinear in the gauge field vanishes, as it did in the former case.

In the noncommutative case there is another contribution, trilinear in the gauge field, similar to equation (37). It contributes with

$$\tilde{I}_{ad}^{(1,3)}[b, A] = \frac{\mathrm{i}}{3} \int \mathrm{d}^4x \sum_{n=1}^{3} \Gamma_{n,ad}^{\mu\rho\nu} [[A_\mu, A'_\rho]_\star, A''_\nu]_\star \Big|_{x''=x'=x} \tag{50}$$

where

$$\Gamma_{n,ad}^{\mu\rho\nu} = \Gamma_n^{\mu\rho\nu} (\partial \times p) \tag{51}$$

and $\Gamma_n^{\mu\rho\nu}$ stands for the three contributions given by equations (38), (39) and (40). These results show that there is no trilinear contribution independent of the derivative.

The above considerations lead to the result that there is no room for Lorentz or CTP violation, despite the representation one chooses for the gauge group. Our results show that there is no UV/IR mixing in the calculation of the induced Chern–Simons term, even when non-planar diagrams are taken into account, as happens in the adjoint representation of the gauge group.

4. Conclusions

We summarize our work recalling that we have calculated the radiative corrections induced by massive and charged Dirac fermions, interacting with an Abelian gauge field and including a non-standard contribution that violates Lorentz and CPT invariance. Our results show that there is an intricate entanglement between infinitely large contributions that come from integration in momentum space, and infinitely small contributions that appear from the trace of Dirac matrices. These two contributions compensate each other, and they do contribute to generate a term that exactly cancels the finite term that appears from the remaining contributions. Because of this intricate cancellation, there is no room for radiative generation of the Chern–Simons-like term. Thus, there is neither Lorentz nor CPT violation generated radiatively. This result is valid under the general guidance of dimensional regularization, despite the spacetime being commutative or noncommutative. In the commutative case, causality leads to a similar result, which excludes the induction of the Chern–Simons-like term [42].

Acknowledgments

We would like to thank CAPES, CNPq, PROCAD and PRONEX for partial support.

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